Lecture 11

Properties of Fourier Transform
(Lathi 7.3-7.4)

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Linearity & Conjugate Properties

- If $x_1(t) \Longleftrightarrow X_1(\omega)$ and $x_2(t) \Longleftrightarrow X_2(\omega)$

  then $a_1 x_1(t) + a_2 x_2(t) \Longleftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$

  Linearity

- If $x(t) \Longleftrightarrow X(\omega)$

  then $x^*(t) \Longleftrightarrow X^*(-\omega)$

  Conjugate

- If $x(t)$ is real

  then $X(-\omega) = X^*(\omega)$

  Conjugate Symmetry

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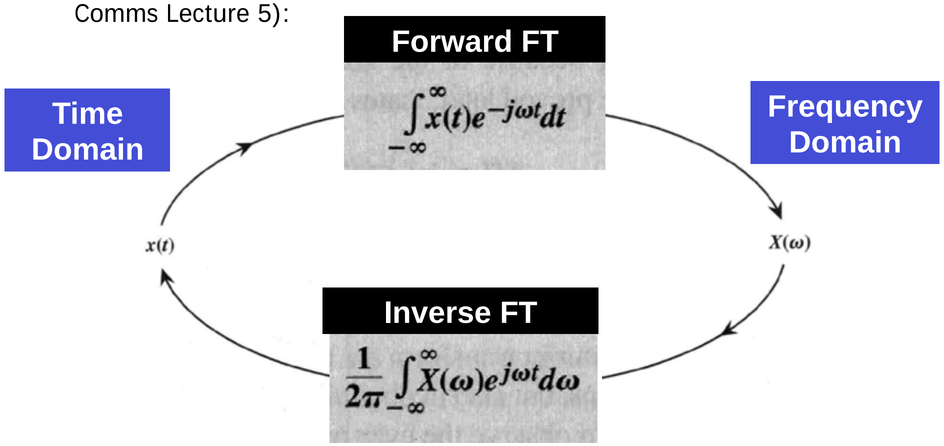
Time-Frequency Duality of Fourier Transform

- Near symmetry between direct and inverse Fourier transforms (Year 1 Comms Lecture 5):



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Duality Property

- If $x(t) \Longleftrightarrow X(\omega)$

  then $X(t) \Longleftrightarrow 2\pi x(-\omega)$

- Proof: From definition of inverse FT (previous slide), we get

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)e^{jut}\,du$$

Hence

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(u)e^{-jut}\,du$$

- Change t to ω yield, and use definition of forward FT, we get:

$$2\pi x(-\omega) \Longleftrightarrow X(t)$$

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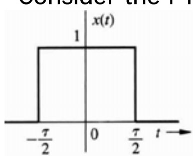
Duality Property Example

- Consider the FT of a rectangular function:



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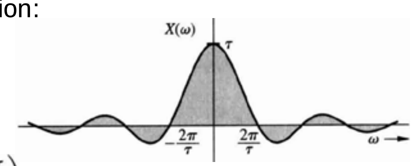
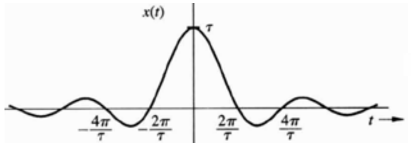


$$\underbrace{\mathrm{rect}\left(\frac{t}{\tau}\right)}_{x(t)} \Longleftrightarrow \underbrace{\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)}_{X(\omega)}$$



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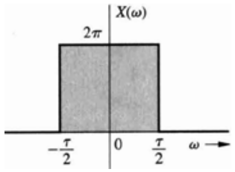


$$\underbrace{\tau\,\mathrm{sinc}\left(\frac{\tau t}{2}\right)}_{X(t)} \Longleftrightarrow \underbrace{2\pi\,\mathrm{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi x(-\omega)} = 2\pi\,\mathrm{rect}\left(\frac{\omega}{\tau}\right)$$

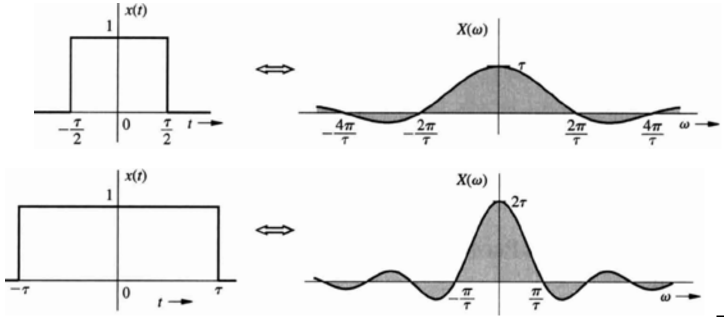
L7.3 p701

Scaling Property

- If $x(t) \Longleftrightarrow X(\omega)$

  then for any real constant a, $x(at) \Longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

- That is, compression of a signal in time results in spectral expansion, and vice versa.



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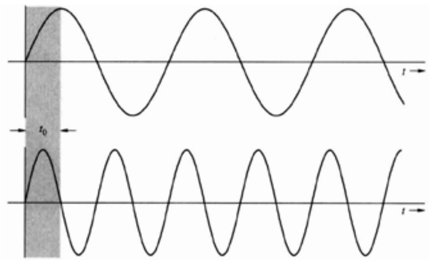
Time-Shifting Property

- If $x(t) \Longleftrightarrow X(\omega)$

  then $x(t - t_0) \Longleftrightarrow X(\omega)e^{-j\omega t_0}$

- Consider a sinusoidal wave, time shifted:



$$\cos\omega(t - t_0) = \cos(\omega t - \omega t_0)$$

- Obvious that phase shift increases with frequency (To is constant).

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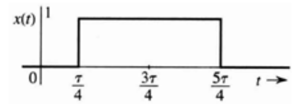
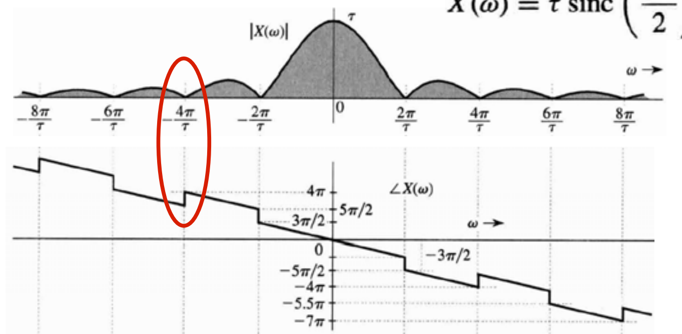
Time-Shifting Example

- Find the Fourier transform of the gate pulse x(t) given by:



- This pulse is rect(t/τ) dleayed by 3τ/4 sec.
- Use time-shifting theorem, we get

$$X(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)e^{-j\omega(3\tau/4)}$$



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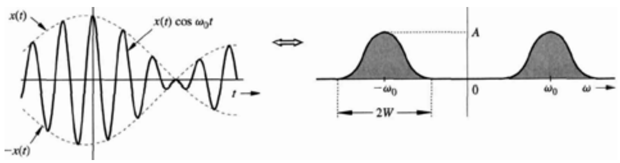
Frequency-Shifting Property

- If $x(t) \Longleftrightarrow X(\omega)$
  then $$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0)$$
- Multiply a signal by $e^{j\omega_0 t}$ shifts the spectrum of the signal by ω_0.
- In practice, frequency shifting (or amplitude modulation) is achieved by multiplying x(t) by a sinusoid:

$$x(t)\cos\omega_0 t = \tfrac{1}{2}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}]$$

$$x(t)\cos\omega_0 t \Longleftrightarrow \tfrac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$$



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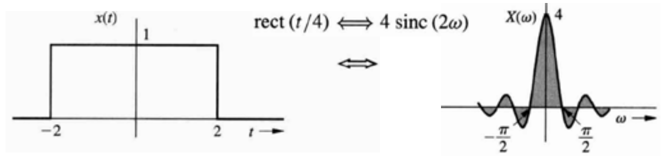
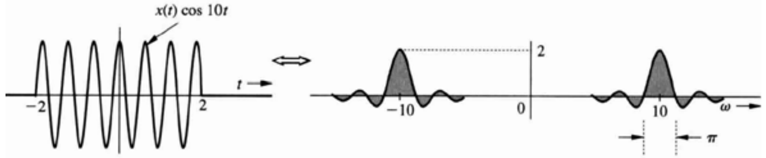
Frequency-Shifting Example

- Find and sketch the Fourier transform of the signal $x(t)\cos 10t$ where $x(t) = rect(t/4)$.



$$x(t)\cos 10t \Longleftrightarrow \tfrac{1}{2}[X(\omega+10) + X(\omega-10)]$$

$$x(t)\cos 10t \Longleftrightarrow 2\,\text{sinc}\,[2(\omega+10)] + 2\,\text{sinc}\,[2(\omega-10)]$$



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Convolution Properties

- If $x_1(t) \Longleftrightarrow X_1(\omega)$ and $x_2(t) \Longleftrightarrow X_2(\omega)$
  then
  $$x_1(t) * x_2(t) \Longleftrightarrow X_1(\omega)X_2(\omega) \quad \text{(time convolution)}$$
  $$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega) \quad \text{(frequency convolution)}$$
- Let H(ω) be the Fourier transform of the unit impulse response h(t), i.e.
  $$h(t) \Longleftrightarrow H(\omega)$$
- Applying the time-convolution property to $y(t)=x(t) * h(t)$, we get:
  $$Y(\omega) = X(\omega)H(\omega)$$
- That is: **the Fourier Transform of the system impulse response is the system Frequency Response**

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Proof of the Time Convolution Properties

- By definition
  $$\mathcal{F}|x_1(t) * x_2(t)| = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau\right]dt$$
  $$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j\omega t}x_2(t-\tau)\,dt\right]d\tau$$
- The inner integral is Fourier transform of $x_2(t-\tau)$, therefore we can use time-shift property and express this as $X_2(\omega)\,e^{-j\omega\tau}$.

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}X_2(\omega)\,d\tau$$
$$= X_2(\omega)\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\,d\tau$$
$$= X_1(\omega)X_2(\omega)$$

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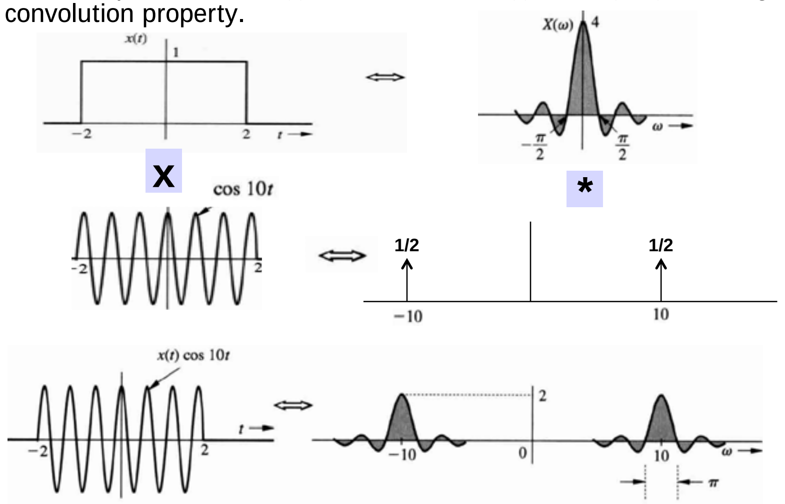
Frequency Convolution Example

- Find the spectrum of $x(t)\cos 10t$ where $x(t) = rect(t/4)$. using convolution property.



Time Differentiation Property

- If $x(t) \Longleftrightarrow X(\omega)$

  then $\frac{dx}{dt} \Longleftrightarrow j\omega X(\omega)$ (time differentiation)

  and $\int_{-\infty}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$ (time integration)

- Compare with Lec 6/17, **Time-differentiation** property of Laplace transform:

$$x(t) \Longleftrightarrow X(s)$$
$$\frac{dx}{dt} \Longleftrightarrow sX(s) - x(0^-)$$

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Summary of Fourier Transform Operations (1)

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$

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Summary of Fourier Transform Operations (2)

Operation	$x(t)$	$X(\omega)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^{t} x(u)\,du$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

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